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# Decomposition of Link Complements(Methods of Transformation Group Theory)

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## Decomposition of Link Complements

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### 1. Introduction

Suppose  $K$  is a knot in  $S^3$ , and  $E(K)$  denotes the exterior of  $K$ . Define a 4-manifold  $M(K)$  to be  $\partial(E(K) \times D^2)$ . This 4-manifold has the same fundamental group as  $E(K)$ , but it is not aspherical. In a talk at the RIMS Conference “Methods of Transformation Group Theory”, May 2006, I announced that the *TOP* surgery obstruction theory works for normal maps to  $M(K)$ . Later I extended the result to the cases of non-split links and non-split subcomplexes of a triangulation. Actually if  $X$  is a connected compact orientable 3-manifold with nonempty boundary such that the assembly map  $A : H_4(X; \mathbb{L}_\bullet) \rightarrow L_4(\pi_1(X))$  is injective, then we have the same conclusion for  $M = \partial(X \times D^2)$ .

Then I learned from Jim Davis that, if the 3-manifold  $X$  is aspherical, the following theorem of Qayum Khan [3] can be applied to these examples to show that the surgery obstruction theory works even in the  $PL = DIFF$  category for normal maps to  $M$ :

**Theorem.** (Khan) *Suppose  $M$  is a closed connected orientable  $PL$  4-manifold with fundamental group  $\pi$  such that the assembly map*

$$A : H_4(\pi; \mathbb{L}_\bullet) \rightarrow L_4(\pi)$$

*is injective, or more generally, the 2-dimensional component of its prime 2 localization*

$$\kappa_2 : H_2(\pi; \mathbb{Z}_2) \rightarrow L_4(\pi)$$

*is injective. Then any degree 1 normal map  $(f, b) : N \rightarrow M$  with vanishing surgery obstruction in  $L_4(\pi)$  is normally bordant to a homotopy equivalence  $M \rightarrow M$ .*

So I decided to change the statement. Let  $X$  be as above.  $X$  has a handle decomposition, and a handle decomposition produces a *CW*-spine  $B$  of  $X$ :  $X$  is a mapping cylinder of some map  $\partial X \rightarrow B$ . The mapping cylinder structure induces a strong deformation retraction  $q : X \rightarrow B$ . Compose this with the projection  $X \times D^2 \rightarrow X$  and restrict it to the boundary to get a map

$p : M = \partial(X \times D^2) \rightarrow B$ . It turns out that, for any choice of the spine  $B$ , this map  $p : M \rightarrow B$  is  $UV^1$  (see [4] for the definition of  $UV^1$ -maps). So the following observation of Hegenbarth and Repovš [2] based on [5] can be applied to  $p : M \rightarrow B$ , if the assembly map is injective.

**Theorem.** (Hegenbarth-Repovš) *Let  $M$  be a closed oriented TOP 4-manifold and  $p : M \rightarrow B$  be a  $UV^1$ -map to a finite CW-complex such that the assembly map*

$$A : H_4(B; \mathbb{L}_\bullet) \rightarrow L_4(\pi_1(B))$$

*is injective. Then the following holds: if  $(f, b) : N \rightarrow M$  is a degree 1 TOP normal map with trivial surgery obstruction in  $L_4(\pi_1(M))$ , then  $(f, b)$  is TOP normally bordant to a  $p^{-1}(\epsilon)$ -homotopy equivalence  $f' : N' \rightarrow M$  for any  $\epsilon > 0$ . In particular  $(f, b)$  is TOP normally bordant to a homotopy equivalence.*

For example, we have

**Theorem.** *If  $X$  is a compact connected orientable Haken 3-manifold with boundary, and  $B$  is any CW-spine of  $X$ , then there is a  $UV^1$ -map  $p : M(X) \rightarrow B$ , and the assembly map  $A : H_4(B; \mathbb{L}_\bullet) \rightarrow L_4(\pi_1(B))$  is an isomorphism. Therefore, if  $(f, b) : N \rightarrow M$  is a degree 1 TOP normal map with trivial surgery obstruction in  $L_4(\pi_1(M))$ , then  $(f, b)$  is TOP normally bordant to a  $p^{-1}(\epsilon)$ -homotopy equivalence  $f' : N' \rightarrow M$  for any  $\epsilon > 0$ .*

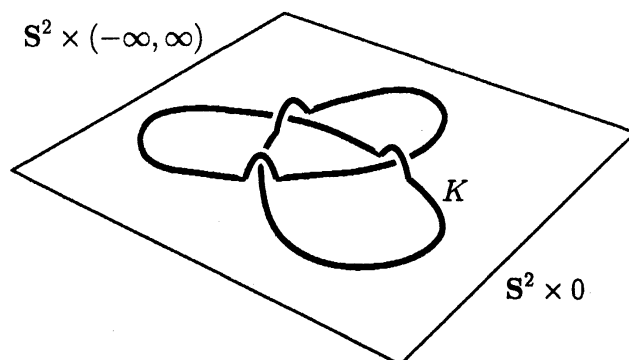
See [8] for details.

In the talk at RIMS, I used an ideal cell decomposition of link complements to construct a spine for  $X = E(K)$ . This is now obsolete. But it may be of some interest, so I will discuss the construction in this note.

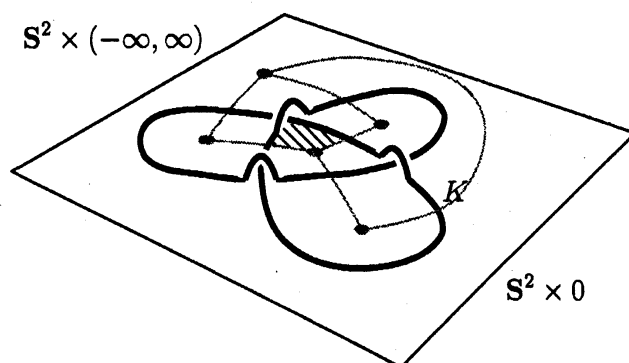
## 2. Ideal Cell Decomposition of Link Complements

Let  $K$  be a knot in  $S^3$ . We show that  $S^3 - K$  decomposes into ideal 3-cells (= 3-cells whose vertices are removed). The following construction works equally well when  $K$  is a link.

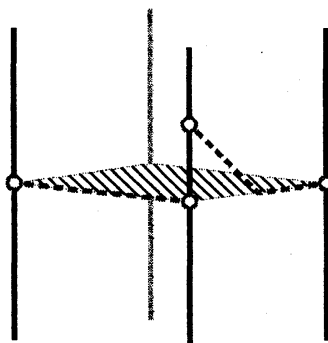
Identify  $S^3$  with  $S^2 \times (-\infty, \infty) \cup \{\pm\infty\}$ , and consider a knot projection to  $S^2 \times 0$ , with  $n$  crossings. We assume that  $n \geq 1$  and that  $K$  stays in  $S^2 \times 0$  except at the overcrossings as in the next picture:



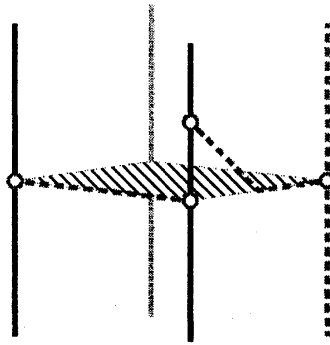
Consider the dual graph of the knot diagram:



The dual graph and the knot diagram together decompose  $S^2 \times 0$  into  $4n$ -many quadrangles  $R_i$ . One such quadrangle is indicated in the picture above. Roughly speaking,  $R_i \times (-\infty, \infty) - K$  are the desired ideal 3-cells:



Unfortunately their union is not  $S^3 - K$ , but  $S^3 - \{\pm\infty\} - K$ . So pick an intersection point of  $K$  and the dual graph, and dig tunnels from that point to  $\pm\infty$  along the edges. This affects four of the 3-cells as in the picture below and gives a decomposition of  $S^3 - K$  into ideal cells:



**Remark.** A knot/link complement has a decomposition into ideal tetrahedra. Discussions on this topic can be found in [1][6][7][9], but these are all quite technical.

The dual spine of the ideal cell decomposition can be defined in the following way: Take one point from each 1-cell; the union of these points is the dual spine of the 1-skeleton and there is a collapsing map from the 1-skeleton to the spine. Next, take one point from the interior of each 2-cell, and take the topological join of the point and the spine of the boundary. The union of these joins is the spine of the 2-skeleton. The collapsing map of the 1-skeleton extends to the collapsing map of the 2-skeleton to the spine. Finally, take one point from the interior of each 3-cell, take the join of the point and the spine of the boundary. The union of these joins is the desired spine  $B$ , and the collapsing map of the 2-skeleton extends to a collapsing map  $q : S^3 - K \rightarrow B$ .

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